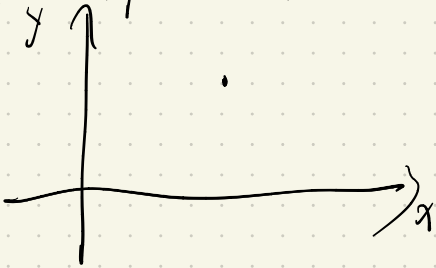
MATH 2230 B 20/01/21.

1. Complex Number



i) $z = x + yi$, $i = \sqrt{-1}$, $x, y \in \mathbb{R}$.

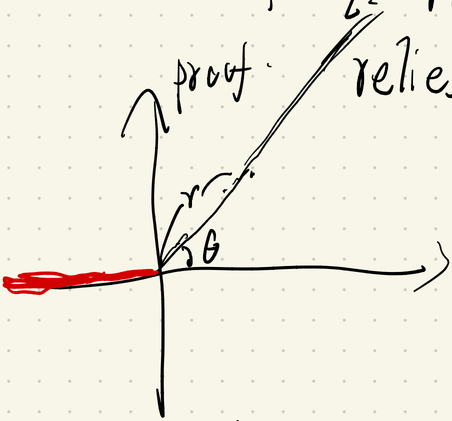
$z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i)$

$= x_1 x_2 + x_1 y_2 i + y_1 x_2 i - y_1 y_2$

$= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2) i$

ii) (r, θ): $z = re^{i\theta}$, $e^{i\theta} = \cos\theta + i\sin\theta$

proof: relies on Taylor expansion.



$\theta + 2n\pi$

$(\alpha, \alpha + 2\pi)$, if $\alpha = -\pi$, $\theta \in [-\pi, \pi)$.

principle argument if $\theta \in [-\pi, \pi)$.

$z = re^{i\theta}$

$(z)^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n \cos n\theta + i r^n \sin n\theta$

$\downarrow$

$(r\cos\theta + ir\sin\theta)^n$

$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + \sin n\theta\, i$

$\Rightarrow$ de Moivre's formula.

For $z = re^{i\theta}$

$\arg(z) = \theta$, argument, multivalued

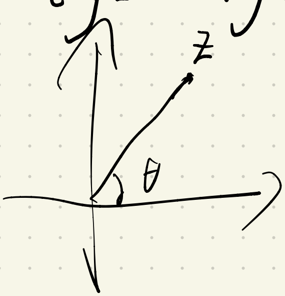
$\operatorname{Arg}(z) = \theta$, principle argument, only one.

2. Logarithm

$f = e^z = e^{x+yi} = e^x e^{yi} = e^x(\cos y + i\sin y)$

$z \in \mathbb{C}$

$\log z = \log re^{i(\theta + 2n\pi)} = \boxed{\ln r} + (\theta + 2n\pi)\, i$



multi-valued.

i) define θ to be in some intervals so that it is well defined.

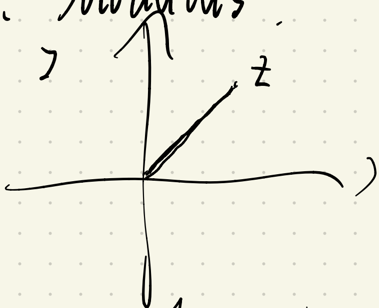
$(\alpha, \alpha+2\pi)$.

If $\alpha = -\pi$, $\theta \in (-\pi, \pi)$, then Log.

$z = re^{i\theta}$, $\theta \in (-\pi, \pi)$

ii) Riemann Surface.

3. Modulus



$z = x + yi$, $|z| = \sqrt{x^2 + y^2}$.

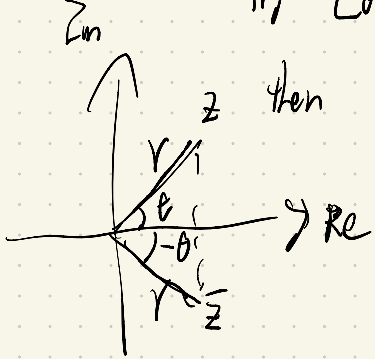
$z = re^{i\theta}$, $|z| = r$.

Triangle Inequality

$|z_1 + z_2| \leq |z_1| + |z_2|$.

Observation: i) $|\mathrm{Re}\, z| \leq |z|$, $|\mathrm{Im}\, z| \leq |z|$

ii) Conjugate of z: If $z = x + yi$



then $\bar{z} = x - yi$,

If $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$.

$z\bar{z} = r^2$.

iii) $z = x+yi$, $\bar{z} = x-yi$

$$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$$

Proof: $|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$

$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$

$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_1|^2 + |z_2|^2 + (z_1\bar{z}_2 + \overline{z_1\bar{z}_2})$$

$$= |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\bar{z}_2)$$

$$\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$$

$$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

$$= (|z_1|+|z_2|)^2$$

$\Rightarrow$ $|z_1+z_2| \leq |z_1|+|z_2|$. $\square$

Rk: i) $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$, $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$.

ii) $|z_1 z_2| = |z_1||z_2|$, can be seen by the exponential representation.

$|e^{i\theta}| = 1$.

$e^{i\theta} = \cos\theta + i\sin\theta$.

4. Limit & Continuity

i) For $z_n \to z$, $\forall \varepsilon > 0$, $\exists N$, s.t $n \geq N$

$|z_n - z| < \varepsilon$,

modulus.

ii) Continuity: If f is a $\mathbb{C}$-valued function, & it is cts at z_0,

$\Rightarrow \forall \varepsilon > 0$, $\exists \delta$, s.t for $\forall z \in \{|z - z_0| < \delta\}$

$|f(z) - f(z_0)| < \varepsilon$.

Rules of limit.

If $w_n \to w$, $z_n \to z$, i) then $\lim(w_n + z_n) = \lim w_n + \lim z_n$

$= w + z$

ii) $\lim w_n z_n = \lim w_n \lim z_n = wz$

iii) If $z \neq 0$, $\lim \frac{w_n}{z_n} = \frac{\lim w_n}{\lim z_n} = \frac{w}{z}$.

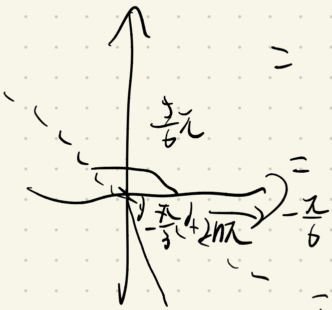
iv) If f is cts, & $z_n \to z$.

Then $\lim f(z_n) = f(z)$.

P30 - 1

b) $z = 1 - \sqrt{3} i$, $|z| = 2$, $\arg z = -\frac{\pi}{3} + \boxed{2n\pi}$

$$z^{\frac{1}{2}} = \left(e^{\log z}\right)^{\frac{1}{2}}$$



$$= e^{\frac{1}{2} \log 2e^{(-\frac{\pi}{3} + 2n\pi) i}}$$

$$= e^{\frac{1}{2}(\ln 2 + (-\frac{\pi}{3} + 2n\pi) i)}$$

$$= \sqrt{2} e^{(-\frac{\pi}{6} + n\pi) i}$$

$$= \begin{cases} \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right), & n \text{ is even.} \\ \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2} i\right), & n \text{ is odd.} \end{cases}$$

$z^c = e^{c \log z}$, never forget $2n\pi i$.

P24 - 9.

$S_n = 1 + z + \cdots + z^n$

$z S_n - S_n = z^{n+1} - 1 \Rightarrow S_n = \frac{1 - z^{n+1}}{1 - z}$ ... (1).

$1 + \cos\theta + \cdots + \cos n\theta \Rightarrow$ plug $z = e^{i\theta}$ into (1)

$LHS = 1 + e^{i\theta} + \cdots + e^{in\theta} \Rightarrow \mathrm{Re}(LHS) = 1 + \cos\theta + \cdots + \cos n\theta$

$RHS = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}$ take Real part of RHS.

Mention: $z_1 z_2$, $\arg(z_1 z_2) = \arg(r_1 r_2 e^{i(\theta_1+\theta_2)})$

$= \theta_1 + \theta_2$

$= \arg z_1 + \arg z_2$

but $\operatorname{Arg}(z_1 z_2) \neq \operatorname{Arg} z_1 + \operatorname{Arg} z_2$.